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Solutions for a time-fractional diffusion equation with absorption: influence of different diffusion coefficients and external forces

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Abstract
This work is devoted to investigate explicit solutions of the time-fractional diffusion equations with external forces by considering various diffusion coefficients and an absorbent rate. Besides, the $2n$th moment related to such an equation is also discussed. Consequently, the diffusion type can be determined from the mean-square displacement. In addition, a rich class of diffusive processes, including normal and anomalous ones, can be obtained.

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1. Introduction

The ubiquity of the anomalous diffusion phenomenon in the nature has attracted the interest of researchers from both the theoretical and experimental point of view. In fact, fractional diffusion equations and the nonlinear fractional diffusion equations have been successfully applied to several physical situations such as percolation of gases through porous media [1], thin saturated regions in porous media [2], in the transport of fluid in porous media and surface growth [3], diffusion of dissolved solutes into immobile water zones of various sizes, a standard solid-on-solid model for surface growth, thin liquid films spreading under gravity [4], modeling of non-Markvian dynamical processes in protein folding [5], relaxation to equilibrium in system (such as polymer chains and membranes) with long temporal memory [6] and anomalous transport in disordered systems [7]. Note that the physical systems mentioned above essentially concern anomalous diffusion of the correlated type (both sub- and super-diffusion; see [8] and references therein) or of the Lévy type (see [9] and references therein). The anomalous correlated diffusion usually has a finite second moment $\langle x^2\rangle \propto t^\sigma$ ($\sigma > 1, \sigma = 1$ and $0 < \sigma < 1$ correspond to super-diffusion, normal diffusion and sub-diffusion, respectively; $\sigma = 0$ corresponds basically to localization). The properties concerning these equations have been intensively investigated [10–14].

In [15], Lenzi *et al* investigated the solutions for the fractional nonlinear diffusion equation

$$\frac{\partial^\gamma}{\partial t^\gamma}P(x,t) = \int_0^t \mathrm{d}t' \frac{\partial}{\partial x}\left\{D(x,t-t')\frac{\partial^{\mu-1}}{\partial x^{\mu-1}}[P(x,t')]^\nu\right\} - \frac{\partial}{\partial x}\{F(x)P(x,t)\}, \qquad 0<\gamma<1, \tag{1}$$

where $\gamma,\mu,\nu \in \mathbb{R}$, $D(x,t) = D(t)|x|^{-\theta}$ is a diffusion coefficient, $F(x) \equiv -\mathrm{d}V(x)/\mathrm{d}x$ is an external force (drift) associated with the potential $V(x)$. In this paper, we will discuss the solutions for the fractional diffusion equation

$$\frac{\partial^\gamma}{\partial t^\gamma}P(x,t) = \int_0^t \mathrm{d}t' \frac{\partial}{\partial x}\left\{D(x,t-t')\frac{\partial^{\mu-1}}{\partial x^{\mu-1}}[P(x,t')]^\nu\right\} - \frac{\partial}{\partial x}\{F(x)P(x,t)\} - \alpha(t)P(x,t), \qquad 0<\gamma<1, \tag{2}$$

with an absorbent or source term $\alpha(t)$, where $\gamma,\mu,\nu \in \mathbb{R}$, $D(x,t) = D(t)$ is a diffusion coefficient, $F(x) \equiv -\mathrm{d}V(x)/\mathrm{d}x$. Here, we use the Caputo operator[1] for the fractional derivative, and we work with the positive spatial variable x. Later on, we will extend the results to the entire real x-axis by the use of symmetry (in other words, we are working with $\partial/\partial|x|$ and $\partial^{\mu-1}/\partial|x|^{\mu-1}$). Also, we employ, in general, the initial condition $P(x,0) = g(x)$ ($g(x)$ is a given function), and the boundary condition $P(x\to\pm\infty,t)\to 0$. Note that when $(\mu,\gamma,\nu) = (2,1,1), \alpha(t) = 0$, equation (2) recovers the standard Fokker–Planck equation in the presence of a drift taking memory effects into account. The particular case $F(x) = 0, \alpha(t) = 0$ and $D(t) = D\delta(t)$ with $(\mu,\gamma) = (2,1)$ has been considered by Spohn [3], and the general case with conditions $(\mu,\gamma) = (2,1)$ and $\alpha(t) = 0$ has been investigated in [17–19]. Our present investigation will be related to the case involving fractional derivatives (i.e. $\gamma \neq 1$) and an absorption term $\alpha(t) = \alpha > 0$.

Explicit solutions play an important role in analyzing physical situations, since they contain, in principle, precise information about the system. In particular, they can be used as a useful guide to control the accuracy of numerical solutions. For these reasons, we dedicated this work to investigate the solutions of a particular case of equation (2) with $0<\gamma<1$, $\mu = 2$, $\nu = 1$, $F(x) = k$, $\alpha(t) = \alpha$, where k and α are positive constants, and a variety of different diffusion coefficients $D(t)$, that is,

$$\frac{\partial^\gamma}{\partial t^\gamma}P(x,t) = \int_0^t \mathrm{d}t' \left\{D(t-t')\frac{\partial^2}{\partial x^2}P(x,t')\right\} - k\frac{\partial}{\partial x}P(x,t) - \alpha P(x,t), \qquad 0<\gamma<1. \tag{3}$$

In order to give support to possible investigations of physical systems modeling by the diffusion equations is studied here. The paper is organized as follows. In section 2, we present the exact solutions of equation (3) with the natural boundary condition $P(x\to\pm\infty,t)\to 0$, the initial condition $P(x,0) = g(x)$ and various diffusion coefficients $D(t)$. In section 3, we analyze the $2n$th moment, including the mean-square displacement, related to equation (3). The conclusion is presented in section 4.

[1] The definition of the fractional Caputo derivative is $\partial^\gamma\rho/\partial t^\gamma = [1/\Gamma(n-\gamma)]\int_0^t \mathrm{d}t'[\rho^{(n)}/(t-t')^{\gamma-n+1}]$, where $n-1<\gamma<n$.

2. Explicit solutions for different $D(t)$

In this section, we focus on equation (3). We first restrict our attention to the positive spatial variable x and then extend to the whole real axis by symmetry. By employing the Laplace transform with respect to the time t in equation (3), we obtain

$$\widetilde{D}(s)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\widetilde{P}(x,s) - k\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{P}(x,s) - (s^\gamma + \alpha)\widetilde{P}(x,s) = -s^{\gamma-1}P(x,0), \qquad (4)$$

where $\widetilde{P}(x,s) = \mathscr{L}\{P(x,t)\}$, $\widetilde{D}(s) = \mathscr{L}\{D(t)\}$ and $\mathscr{L}\{f(t)\} = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-st}f(t)$ denotes the Laplace transform of the function f where $\mathrm{Re}\,s \gg 1$, and the branch of s^γ is taken such that $s^\gamma > 0$ when $s > 0$. In equation (4), use has been made that $\mathscr{L}\{\partial^\gamma P(x,t)/\partial t^\gamma\} = s^\gamma\widetilde{P}(x,s) - s^{\gamma-1}P(x,t)|_{t=0}$ (see [20]). Equation (4) is a linear, non-homogenous, ordinary differential equation with respect to the variable x, which can be solved explicitly. Let us denote $P_\delta(x,t)$ the solution of equation (3) with the initial condition $P(x,0) = \delta(x)$ and $\widetilde{P}_\delta(x,s)$ is the Laplace transform of $P_\delta(x,t)$ with respect to t. Then $\widetilde{P}_\delta(x,s)$ is the solution of equation (4) with $P(x,0) = \delta(x)$ on the right-hand side of (4), which can be solved as

$$\widetilde{P}_\delta(x,s) = C_1\,\mathrm{e}^{\lambda_1 x} + C_2\,\mathrm{e}^{\lambda_2 x} - \frac{s^{\gamma-1}}{\widetilde{D}(s)}\int_0^x \frac{\mathrm{e}^{\lambda_1(x-t)} - \mathrm{e}^{\lambda_2(x-t)}}{\lambda_1 - \lambda_2}\delta(t)\,\mathrm{d}t, \qquad (5)$$

where

$$\lambda_1 = \frac{k + \sqrt{k^2 + 4\widetilde{D}(s)(\alpha + s^\gamma)}}{2\widetilde{D}(s)}, \qquad \lambda_2 = \frac{k - \sqrt{k^2 + 4\widetilde{D}(s)(\alpha + s^\gamma)}}{2\widetilde{D}(s)},$$

and C_1, C_2 are constants to be determined. Since $\int_0^x \mathrm{d}t\,\mathrm{e}^{\lambda_i(x-t)}\delta(t) = \mathrm{e}^{\lambda_i x}$, we have

$$\widetilde{P}_\delta(x,s) = \left[C_1 - \frac{s^{\gamma-1}}{\widetilde{D}(s)(\lambda_1 - \lambda_2)}\right]\mathrm{e}^{\lambda_1 x} + \left[C_2 + \frac{s^{\gamma-1}}{\widetilde{D}(s)(\lambda_1 - \lambda_2)}\right]\mathrm{e}^{\lambda_2 x}. \qquad (6)$$

To determine the constants C_1, C_2, we restrict s to be a sufficiently large positive number. Assuming $\widetilde{D}(s) > 0$, which is the case we discussed below, we have $\lambda_1 > 0, \lambda_2 < 0$. Note that the boundary condition $P(x,t) \to 0 (x \to +\infty)$ implies that $\widetilde{P}_\delta(x,s) \to 0 (x \to +\infty)$, and the normalization condition $\int_{-\infty}^{+\infty}\mathrm{d}x\,P(x,t) = 1$, implies that $\int_0^{+\infty}\mathrm{d}x\,\widetilde{P}(x,s) = 1/2s$. We finally obtain the solution of equation (4) with $P(x,0) = \delta(x)$:

$$\widetilde{P}_\delta(x,s) = -\frac{\lambda_2}{2s}\,\mathrm{e}^{\lambda_2|x|} = \frac{-k + \sqrt{k^2 + 4\widetilde{D}(s)(s^\gamma + \alpha)}}{4\widetilde{D}(s)s}\,\mathrm{e}^{\frac{k - \sqrt{k^2 + 4\widetilde{D}(s)(s^\gamma + \alpha)}}{2\widetilde{D}(s)}|x|}. \qquad (7)$$

Now, the solution $P_\delta(x,t)$ of equation (3) is the inverse Laplace transform of $\widetilde{P}_\delta(x,s)$ given in (7), which is difficult to get an explicit formula in general. For some spacial parameters and diffusion coefficient $D(t)$, it can be expressed in Fox H-function. For arbitrary $0 < \gamma < 1$, we prefer to get a power series solution for $P_\delta(x,t)$ by using Taylor's expansions of the exponential function and the binomial function:

$$\mathrm{e}^z = \sum_{n=0}^{+\infty}\frac{z^n}{n!}, \quad \text{and} \quad (1+z)^\alpha = 1 + \alpha z + C_\alpha^2 z^2 + C_\alpha^3 z^3 + \cdots + C_\alpha^n z^n + \cdots, \qquad |z| < 1,$$

where $C_n^\alpha = \alpha(\alpha - 1)\cdots(\alpha - n + 1)/n!$.

Now we derive the explicit solution of equation (3) by making the inverse Laplace transform of $\widetilde{P}_\delta(x,s)$ for various diffusion coefficients:

Case 1. $D(t) = D\delta(t)$, i.e. $\widetilde{D}(s) = D$, the case which was also discussed in [8, 19]. When $\mathrm{Re}\,s \gg 1$, we have the following expansions:

$$\widetilde{P}_\delta(x,s) = \sum_{n=0}^{+\infty}\frac{(-1)|x|^n(k - \sqrt{k^2 + 4D(\alpha + s^\gamma)})^{n+1}}{n!2^{n+2}D^{n+1}s},$$

$$\left(1-\frac{k}{\sqrt{k^2+4D(\alpha+s^\gamma)}}\right)^{n+1}=\sum_{m=0}^{n+1}C_{n+1}^m(-1)^m k^m(k^2+4D(\alpha+s^\gamma))^{-m/2},$$

$$(k^2+4D(\alpha+s^\gamma))^{(n+1-m)/2}=\sum_{j=0}^{+\infty}C_{(n+1-m)/2}^j k^{2j}(4D(\alpha+s^\gamma))^{(n+1-m)/2-j},$$

and

$$(\alpha+s^\gamma)^{(n+1-m)/2-j}=\sum_{i=0}^{+\infty}C_{(n+1-m)/2-j}^i\alpha^i s^{\gamma[(n+1-m)/2-j-i]}.$$

Inserting these into (7), we have

$$\begin{aligned}\widetilde{P}_\delta(x,s)=&\sum_{n=0}^{\infty}\frac{|x|^n}{n!}\frac{(-1)}{2^{n+2}}\sum_{m=0}^{n+1}C_{n+1}^m(-1)^{n+m+1}k^m\sum_{j=0}^{\infty}C_{(n+1-m)/2}^j k^{2j}2^{n+1-m-2j}\\&\times\sum_{i=0}^{\infty}C_{(n+1-m)/2-j}^i\alpha^i\frac{s^{\gamma[(n+m+1)/2-j-i]-1}}{D^{(n+m+1)/2+j}}.\end{aligned}\tag{8}$$

By applying the known Laplace transform of the function t^v [21, formula 4.3(1)], due to the uniqueness of the Laplace transform, we obtain the explicit solution to equation (3) herein:

$$\begin{aligned}P_\delta(x,t)=&\sum_{n=0}^{\infty}\frac{|x|^n}{n!}\frac{(-1)}{2^{n+2}}\sum_{m=0}^{n+1}C_{n+1}^m(-1)^{n+m+1}k^m\sum_{j=0}^{\infty}C_{(n+1-m)/2}^j k^{2j}2^{n+1-m-2j}\\&\times\sum_{i=0}^{\infty}C_{(n+1-m)/2-j}^i\alpha^i\frac{t^{-\gamma[(n+m+1)/2-j-i]}}{D^{(n+m+1)/2+j}\Gamma\left(1-\gamma\left(\frac{n+m+1}{2}-j-i\right)\right)}.\end{aligned}\tag{9}$$

In particular, if $k=0$, i.e. without the external force, and $\alpha\neq 0$, equation (9) becomes

$$P_\delta(x,t)=\sum_{n=0}^{\infty}\frac{|x|^n}{n!}\frac{1}{2}\left(\frac{1}{D}\right)^{\frac{n+1}{2}}\sum_{i=0}^{\infty}C_{(n+1)/2}^i\alpha^i\frac{t^{-\gamma(n+1)/2-i}}{\Gamma\left(1-\gamma\left(\frac{n+1}{2}-i\right)\right)}.\tag{10}$$

Similarly, if $k\neq 0$, $\alpha=0$, equation (9) reduces to

$$\begin{aligned}P_\delta(x,t)=&\sum_{n=0}^{\infty}\frac{|x|^n}{n!}\frac{(-1)}{2^{n+2}}\sum_{m=0}^{n+1}C_{n+1}^m(-1)^{n+m+1}k^m\\&\times\sum_{j=0}^{\infty}C_{(n+1-m)/2}^j k^{2j}2^{n+1-m-2j}\frac{t^{-\gamma[(n+m+1)/2-j]}}{D^{(n+m+1)/2+j}\Gamma\left(1-\gamma\left(\frac{n+m+1}{2}-j\right)\right)}.\end{aligned}\tag{11}$$

When $k=0$, $\alpha=0$, we recover the result given in [19]:

$$P_\delta(x,t)=\sum_{n=0}^{\infty}\frac{|x|^n}{n!}\frac{1}{2}\left(\frac{1}{D}\right)^{\frac{n+1}{2}}\frac{t^{-\gamma(n+1)/2}}{\Gamma\left(1-\gamma\left(\frac{n+1}{2}\right)\right)}.\tag{12}$$

From (9), the solution of equation (3), subject to the general initial condition $P(x,0)=g(x)$, is given by

$$\begin{aligned}P_g(x,t)&=\int_{-\infty}^{+\infty}\mathrm{d}x'g(x-x')P_\delta(x',t)\\&=\int_{-\infty}^{+\infty}\mathrm{d}x'g(x-x')\left\{\sum_{n=0}^{\infty}\frac{|x'|^n}{n!}\frac{(-1)}{2^{n+2}}\sum_{m=0}^{n+1}C_{n+1}^m(-1)^{n+m+1}k^m\right.\end{aligned}$$

$$\times \sum_{j=0}^{\infty} C_{(n+1-m)/2}^{j} k^{2j} 2^{n+1-m-2j}$$
$$\times \sum_{i=0}^{\infty} C_{(n+1-m)/2-j}^{i} \alpha^{i} \frac{t^{-\gamma[(n+m+1)/2-j-i]}}{D^{(n+m+1)/2+j}\Gamma\left(1-\gamma\left(\frac{n+m+1}{2}-j-i\right)\right)}\Bigg\}, \tag{13}$$

where $P_g(x,t)$ denotes the solution of equation (3) subjects to the initial condition $P(x,0) = g(x)$.

Case 2. $D(t) = D\,\mathrm{e}^{-t/\tau_c}/\tau_c$, i.e. $\widetilde{D}(s) = D/(1+\tau_c s)$, the Cattaneo case [18]. By the similar way we discussed in Case 1, we get from equation (7) that

$$P_\delta(x,t) = \sum_{n=0}^{\infty} \frac{|x|^n}{n!} \frac{(-1)}{2^{n+2}} \sum_{m=0}^{n+1} C_{n+1}^{m} (-1)^{n+m+1} k^m \sum_{j=0}^{\infty} C_{(n+1-m)/2}^{j} k^{2j} 2^{n+1-m-2j}$$
$$\times \sum_{i=0}^{\infty} C_{(n+1-m)/2-j}^{i} \alpha^{i} \left(\frac{\tau_c}{D}\right)^{\frac{n+m+1}{2}+j} \sum_{l=0}^{\infty} C_{(n+m+1)/2+j}^{l}$$
$$\times \frac{1}{\tau_c^l} \frac{t^{l-\gamma[(n+m+1)/2-j-i]-(n+m+1)/2-j}}{\Gamma\left(1+l-\gamma\left(\frac{n+m+1}{2}-j-i\right)-\frac{n+m+1}{2}-j\right)}. \tag{14}$$

In particular, if $k=0$, $\alpha \neq 0$, we obtain from equation (14) that

$$P_\delta(x,t) = \sum_{n=0}^{\infty} \frac{|x|^n}{n!} \frac{1}{2} \left(\frac{\tau_c}{D}\right)^{\frac{n+1}{2}} \sum_{i=0}^{\infty} C_{(n+1)/2}^{i} \alpha^{i}$$
$$\times \sum_{l=0}^{\infty} C_{(n+1)/2}^{l} \frac{1}{\tau_c^l} \frac{t^{l-\gamma[(n+1)/2-i]-(n+1)/2}}{\Gamma\left(1+l-\gamma\left(\frac{n+1}{2}-i\right)-\frac{n+1}{2}\right)}; \tag{15}$$

if $k \neq 0$, $\alpha = 0$, we have

$$P_\delta(x,t) = \sum_{n=0}^{\infty} \frac{|x|^n}{n!} \frac{(-1)}{2^{n+2}} \sum_{m=0}^{n+1} C_{n+1}^{m} (-1)^{n+m+1} k^m \sum_{j=0}^{\infty} C_{(n+1-m)/2}^{j} k^{2j} 2^{n+1-m-2j}$$
$$\times \left(\frac{\tau_c}{D}\right)^{\frac{n+m+1}{2}+j} \sum_{l=0}^{\infty} C_{(n+m+1)/2+j}^{l} \frac{1}{\tau_c^l} \frac{t^{l-\gamma[(n+m+1)/2-j]-(n+m+1)/2-j}}{\Gamma\left(1+l-\gamma\left(\frac{n+m+1}{2}-j\right)-\frac{n+m+1}{2}-j\right)}; \tag{16}$$

and if $k=0$, $\alpha=0$, we recover the result given in [18]

$$P_\delta(x,t) = \sum_{n=0}^{\infty} \frac{|x|^n}{n!} \frac{1}{2} \left(\frac{\tau_c}{D}\right)^{\frac{n+1}{2}} \sum_{l=0}^{\infty} C_{(n+1)/2}^{l} \frac{1}{\tau_c^l} \frac{t^{l-(\gamma+1)\frac{n+1}{2}}}{\Gamma\left(1+l-(\gamma+1)\frac{n+1}{2}\right)}. \tag{17}$$

From (14), the solution of equation (3), subject to the general initial condition $P(x,0) = g(x)$, is given by

$$P_g(x,t) = \int_{-\infty}^{+\infty} \mathrm{d}x' g(x-x') P_\delta(x',t)$$
$$= \int_{-\infty}^{+\infty} \mathrm{d}x' g(x-x') \Bigg\{ \sum_{n=0}^{\infty} \frac{|x'|^n}{n!} \frac{(-1)}{2^{n+2}} \sum_{m=0}^{n+1} C_{n+1}^{m} (-1)^{n+m+1} k^m$$
$$\times \sum_{j=0}^{\infty} C_{(n+1-m)/2}^{j} k^{2j} 2^{n+1-m-2j} \sum_{i=0}^{\infty} C_{(n+1-m)/2-j}^{i} \alpha^{i} \left(\frac{\tau_c}{D}\right)^{\frac{n+m+1}{2}+j} \sum_{l=0}^{\infty} C_{(n+m+1)/2+j}^{l}$$
$$\times \frac{1}{\tau_c^l} \frac{t^{l-\gamma[(n+m+1)/2-j-i]-(n+m+1)/2-j}}{\Gamma\left(1+l-\gamma\left(\frac{n+m+1}{2}-j-i\right)-\frac{n+m+1}{2}-j\right)} \Bigg\}. \tag{18}$$

Case 3. $D(t) = Dt^{a-1}/\Gamma(a), a > 0$, i.e. $\widetilde{D}(s) = Ds^{-a}$. In this case, we get from equation (7) that

$$P_\delta(x,t) = \sum_{n=0}^{\infty}\frac{|x|^n}{n!}\frac{(-1)}{2^{n+2}}\sum_{m=0}^{n+1}C_{n+1}^m(-1)^{n+m+1}k^m\sum_{j=0}^{\infty}C_{(n+1-m)/2}^j k^{2j}2^{n+1-m-2j}$$
$$\times\sum_{i=0}^{\infty}C_{(n+1-m)/2-j}^i\alpha^i\frac{t^{-\gamma[(n+m+1)/2-j-i]-a[(n+m+1)/2+j]}}{D^{(n+m+1)/2+j}\Gamma\left(1-\gamma\left(\frac{n+m+1}{2}-j-i\right)-a\left(\frac{n+m+1}{2}+j\right)\right)}. \tag{19}$$

In particular, if $k = 0, \alpha \neq 0$, from equation (19), we have

$$P_\delta(x,t) = \sum_{n=0}^{\infty}\frac{|x|^n}{n!}\frac{1}{2}\left(\frac{1}{D}\right)^{\frac{n+1}{2}}\sum_{i=0}^{\infty}C_{(n+1)/2}^i\alpha^i\frac{t^{-\gamma[(n+1)/2-i]-a(\frac{n+1}{2})}}{\Gamma\left(1-\gamma\left(\frac{n+1}{2}-i\right)-a\left(\frac{n+1}{2}\right)\right)}. \tag{20}$$

Moreover, if $k \neq 0, \alpha = 0$, we have

$$P_\delta(x,t) = \sum_{n=0}^{\infty}\frac{|x|^n}{n!}\frac{(-1)}{2^{n+2}}\sum_{m=0}^{n+1}C_{n+1}^m(-1)^{n+m+1}k^m\sum_{j=0}^{\infty}C_{(n+1-m)/2}^j k^{2j}2^{n+1-m-2j}$$
$$\times\frac{t^{-\gamma[(n+m+1)/2-j]-a(\frac{n+m+1}{2}+j)}}{D^{(n+m+1)/2+j}\Gamma\left(1-\gamma\left(\frac{n+m+1}{2}-j\right)-a\left(\frac{n+m+1}{2}+j\right)\right)}. \tag{21}$$

If $k = 0, \alpha = 0$, equation (19) reduces to the result given in [15]

$$P_\delta(x,t) = \sum_{n=0}^{\infty}\frac{|x|^n}{n!}\frac{1}{2}\left(\frac{1}{D}\right)^{\frac{n+1}{2}}\frac{t^{-(\gamma+a)(n+1)/2}}{\Gamma\left(1-(\gamma+a)\left(\frac{n+1}{2}\right)\right)}. \tag{22}$$

From (19), the solution of equation (3), subject to the general initial condition $P(x,0) = g(x)$, is given by

$$P_g(x,t) = \int_{-\infty}^{+\infty}\mathrm{d}x' g(x-x')P_\delta(x',t)$$
$$= \int_{-\infty}^{+\infty}\mathrm{d}x' g(x-x')\left\{\sum_{n=0}^{\infty}\frac{|x'|^n}{n!}\frac{(-1)}{2^{n+2}}\sum_{m=0}^{n+1}C_{n+1}^m(-1)^{n+m+1}k^m\right.$$
$$\times\sum_{j=0}^{\infty}C_{(n+1-m)/2}^j k^{2j}2^{n+1-m-2j}$$
$$\left.\times\sum_{i=0}^{\infty}C_{(n+1-m)/2-j}^i\alpha^i\frac{t^{-\gamma[(n+m+1)/2-j-i]-a(\frac{n+m+1}{2}+j)}}{D^{(n+m+1)/2+j}\Gamma\left(1-\gamma\left(\frac{n+m+1}{2}-j-i\right)-a\left(\frac{n+m+1}{2}+j\right)\right)}\right\}. \tag{23}$$

3. Mean square displacement

In order to investigate the properties of the solutions obtained in section 2, corresponding to equation (3). We explore formal results for the time behavior of the moments, in particular, of the mean-square displacement. In this direction, by simple calculation, we obtain a dynamic equation for the $2n$th moment:

$$\frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma}\langle x^{2n}\rangle(t) = 2n(2n-1)\int_0^t\mathrm{d}t' D(t-t')\langle x^{2n-2}\rangle(t') - \alpha\langle x^{2n}\rangle(t) \tag{24}$$

for $n = 1, 2, \ldots$. To find the $2n$th moment related to this equation, a coupled system of equations needs to be solved.

Remark. Because $P(x,t)$ is an even function, we can easily get $\langle x^{2n-1}\rangle(t) = \int_{-\infty}^{+\infty} x^{2n-1} P(x,t)\,\mathrm{d}x = 0$. That is why we just study the $2n$th moment corresponding to equation (3).

With the initial condition $P(x,0) = \delta(x)$, by employing the Laplace transform in equation (24), we obtain that

$$s^{\gamma}\langle\widetilde{x}^{2n}\rangle - s^{\gamma-1}\langle x^{2n}\rangle|_{t=0} = 2n(2n-1)\widetilde{D}(s)\langle\widetilde{x}^{2n-2}\rangle - \alpha\langle\widetilde{x}^{2n}\rangle, \tag{25}$$

where $\langle\widetilde{x}^{2n}\rangle$ denotes the Laplace transform of $\langle x^{2n}\rangle$. Since $\langle x^{2n}\rangle|_{t=0} = \int_{-\infty}^{+\infty} x^{2n} P(x,0)\,\mathrm{d}x = \int_{-\infty}^{+\infty} x^{2n}\delta(x)\,\mathrm{d}x = 0$, we get a recurrent relation

$$\langle\widetilde{x}^{2n}\rangle = 2n(2n-1)\frac{\widetilde{D}(s)}{s^{\gamma}+\alpha}\langle\widetilde{x}^{2n-2}\rangle. \tag{26}$$

Equation (26) can be written as a coupled system of equations:

$$\begin{cases} \langle\widetilde{x}^{2n}\rangle = 2n(2n-1)\frac{\widetilde{D}(s)}{s^{\gamma}+\alpha}\langle\widetilde{x}^{2n-2}\rangle, \\ \langle\widetilde{x}^{2n-2}\rangle = (2n-2)(2n-3)\frac{\widetilde{D}(s)}{s^{\gamma}+\alpha}\langle\widetilde{x}^{2n-4}\rangle, \\ \qquad\qquad \cdots\cdots\cdots\cdots, \\ \langle\widetilde{x}^{2}\rangle = 2\frac{\widetilde{D}(s)}{s^{\gamma}+\alpha}\frac{1}{s}, \end{cases}$$

here, we have used the fact $\langle\widetilde{x}^{0}\rangle = 1/s$. Taking the multiplication on both sides of the above system concludes that

$$\langle\widetilde{x}^{2n}\rangle = \frac{(2n)!}{s}\left(\frac{\widetilde{D}(s)}{s^{\gamma}+\alpha}\right)^{n}. \tag{27}$$

In the following, we take the inverse Laplace transform on both side of equation (27) for various diffusion coefficients in time.

Case 1. $D(t) = D\delta(t)$, i.e. $\widetilde{D}(s) = D$. In this case,

$$\langle\widetilde{x}^{2n}\rangle = \frac{(2n)!}{s}D^{n}\left(\frac{1}{s^{\gamma}+\alpha}\right)^{n}. \tag{28}$$

By using the Laplace transform of the generalized Mittag–Leffler function given in [20]

$$\int_0^{\infty} \mathrm{e}^{-st} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm a t^{\alpha})\,\mathrm{d}t = \frac{k!\,s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}} \qquad (\mathrm{Re}(s) > |a|^{1/\alpha}), \tag{29}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag–Leffler function defined by $E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} z^n/\Gamma(n\alpha+\beta)$, and $E_{\alpha,\beta}^{(k)}(z) = d^k E_{\alpha,\beta}(z)/dz^k$, we have

$$\langle x^{2n}\rangle = \frac{(2n)!}{(n-1)!}D^{n} t^{n\gamma} E_{\gamma,\gamma+1}^{(n-1)}(-\alpha t^{\gamma}). \tag{30}$$

By taking $n = 1$ in equation (30), we obtain the mean-square displacement

$$\langle x^{2}\rangle = 2Dt^{\gamma}E_{\gamma,\gamma+1}(-\alpha t^{\gamma}). \tag{31}$$

From equation (31), we can obtain that $\langle x^2\rangle \approx \frac{2D}{\Gamma(1+\gamma)}\left[1 - \alpha\frac{\Gamma(\gamma+1)}{\Gamma(2\gamma+1)}t^{\gamma}\right]t^{\gamma}$ while $t \ll 1$, which implies that for $0 < \gamma < 1$, the system is a sub-diffusion for $t \ll 1$, and that the generalized diffusion coefficient $K_{\gamma} = D\left[1 - \alpha\frac{\Gamma(\gamma+1)}{\Gamma(2\gamma+1)}t^{\gamma}\right]$ is not a constant but a function of time t if $\alpha > 0$.

To get the behavior of $\langle x^2\rangle$ for $t \gg 1$, we use the asymptotic formula (see [22])

$$E_{\gamma,\beta}(z) \sim -\frac{1}{\Gamma(\beta-\gamma)}\frac{1}{z}, \qquad z \to -\infty, \tag{32}$$

for $0 < \gamma < 1$. When $\alpha > 0$, we get

$$\langle x^2\rangle = 2Dt^{\gamma}E_{\gamma,\gamma+1}(-\alpha t^{\gamma}) \approx \frac{2D}{\alpha}, \qquad t \gg 1. \tag{33}$$

This shows that the mean-square displacement of the system is independent of the parameter γ and has zero diffusion exponent for large time, which corresponds basically to localization.

When $\alpha = 0$, equation (31) reduces to the result given in [15]

$$\langle x^2\rangle = \frac{2D}{\Gamma(1+\gamma)}t^{\gamma}, \tag{34}$$

which implies that for $0 < \gamma < 1$, the system is a sub-diffusion.

Case 2. $D(t) = D\,\mathrm{e}^{-t/\tau_c}/\tau_c$, i.e. $\widetilde{D}(s) = D/(1+\tau_c s)$. We have

$$\langle \widetilde{x}^{2n}\rangle = \frac{(2n)!}{s}\left(\frac{D}{\tau_c}\right)^n\left(\frac{1}{s^{\gamma}+\alpha}\right)^n\left(\frac{1}{s+1/\tau_c}\right)^n. \tag{35}$$

Applying the Laplace transform of the Mittag–Leffler function equation (29) to equation (35) gives

$$\langle x^{2n}\rangle = \frac{(2n)!}{[(n-1)!]^2}\left(\frac{D}{\tau_c}\right)^n\int_0^t \mathrm{d}t'(t-t')^{n\gamma-1}E_{1,1}^{(n-1)}\left(-\frac{1}{\tau_c}(t-t')\right)t'^{n\gamma}E_{\gamma,\gamma+1}^{(n-1)}(-\alpha t'^{\gamma}). \tag{36}$$

Taking $n = 1$ in equation (36) yields the mean-square displacement

$$\langle x^2\rangle = \frac{2D}{\tau_c}\int_0^t \mathrm{d}t'(t-t')^{\gamma-1}E_{1,1}\left(-\frac{1}{\tau_c}(t-t')\right)t'^{\gamma}E_{\gamma,\gamma+1}(-\alpha t'^{\gamma}). \tag{37}$$

Thus, we can obtain that

$$\langle x^2\rangle \approx \frac{2}{\Gamma(1+2\gamma)}\frac{D\Gamma(\gamma)}{\tau_c}\left[1-\alpha\frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}t^{\gamma}\right]t^{2\gamma} \tag{38}$$

for $t \ll 1$, and that the generalized diffusion coefficient $K_{2\gamma} = \frac{D\Gamma(\gamma)}{\tau_c}\left[1-\alpha\frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}t^{\gamma}\right]$ is not a constant but a function of time t, if $\alpha > 0$.

For $t \gg 1$, we can also get for $\alpha > 0$

$$\langle x^2\rangle \approx \frac{2D}{\alpha\tau_c}\int_0^t \mathrm{d}t'(t-t')^{\gamma-1}E_{1,1}\left(-\frac{1}{\tau_c}(t-t')\right) \approx \frac{2D\Gamma(\gamma)\tau_c^{\gamma-1}}{\alpha}, \tag{39}$$

which also has zero diffusion exponent for large time, and corresponds basically to localization.

When $\alpha = 0$, equation (37) becomes

$$\begin{aligned}\langle x^2\rangle &= \frac{2D}{\tau_c\Gamma(1+\gamma)}\int_0^t \mathrm{d}t'(t-t')^{\gamma-1}E_{1,1}\left(-\frac{1}{\tau_c}(t-t')\right)t'^{\gamma}\\ &= \frac{2D}{\tau_c\Gamma(1+\gamma)}\int_0^t \mathrm{d}t'\,t'^{\gamma-1}(t-t')^{\gamma}\,\mathrm{e}^{-t'/\tau_c}.\end{aligned} \tag{40}$$

For $t \ll 1$, by setting $\alpha = 0$ in equation (38), we can obtain that $\langle x^2\rangle \approx \frac{2}{\Gamma(1+2\gamma)}\frac{D\Gamma(\gamma)}{\tau_c}t^{2\gamma}$, which means that the system is a sub-diffusion if $0 < \gamma < \frac{1}{2}$, a normal diffusion if $\gamma = \frac{1}{2}$, and a super-diffusion if $\frac{1}{2} < \gamma < 1$. For $t \gg 1$, it follows from Watson's Lemma [23], $\langle x^2 \rangle \approx \frac{2D\tau_c^{\gamma-1}}{\gamma} t^\gamma$, which corresponds to a sub-diffusion for $0 < \gamma < 1$.

Case 3. $D(t) = D\frac{t^{a-1}}{\Gamma(a)}$, $a > 0$, i.e. $\widetilde{D}(s) = Ds^{-a}$. We have

$$\langle \widetilde{x}^{2n} \rangle = (2n)! \frac{D^n}{s^{1+an}} \left(\frac{1}{s^\gamma + \alpha} \right)^n. \tag{41}$$

Applying equation (29) in equation (41) gives that

$$\langle x^{2n} \rangle = \frac{(2n)!}{(n-1)!} D^n t^{n(\gamma+a)} E^{(n-1)}_{\gamma,\gamma+a+1}(-\alpha t^\gamma). \tag{42}$$

By taking $n = 1$ in equation (42), we obtain the mean-square displacement

$$\langle x^2 \rangle = 2Dt^{\gamma+a} E_{\gamma,\gamma+a+1}(-\alpha t^\gamma). \tag{43}$$

For $t \ll 1$, equation (43) leads to

$$\langle x^2 \rangle \approx \frac{2D}{\Gamma(1+\gamma+a)} \left[1 - \alpha \frac{\Gamma(1+\gamma+a)}{\Gamma(1+2\gamma+a)} t^\gamma \right] t^{\gamma+a}. \tag{44}$$

And that the generalized diffusion coefficient $K_{\gamma+a} = D\big[1 - \alpha \frac{\Gamma(1+\gamma+a)}{\Gamma(1+2\gamma+a)} t^\gamma\big]$ is not a constant but a function of time t.

For $t \gg 1$, we also get from (32)

$$\langle x^2 \rangle = 2Dt^{\gamma+a} E_{\gamma,\gamma+a+1}(-\alpha t^\gamma) \approx \frac{2D}{\alpha \Gamma(a+1)} t^a. \tag{45}$$

This shows that the system is a sub-diffusion, normal diffusion and super-diffusion, when $0 < a < 1$, $a = 1$ and $a > 1$, respectively. Furthermore, it is independent of γ.

Let $\alpha = 0$ in equation (43). We recover the result given in [15]

$$\langle x^2 \rangle = \frac{2D}{\Gamma(1+\gamma+a)} t^{\gamma+a}. \tag{46}$$

It shows that $0 < a + \gamma < 1$, $a + \gamma = 1$ and $a + \gamma > 1$ correspond to sub-diffusion, normal diffusion and super-diffusion.

4. Summary and conclusions

In summary, we have worked on a one-dimensional generalized fractional diffusion equation (3). In this context, we obtained the explicit solutions for the probability density which satisfies the fractional diffusion equation (3) for various diffusion coefficients $D(t)$ with constant external forces and absorbent. When the absorbent disappeared, we recover the results previously obtained by other authors. We also obtained the expressions of $2n$th moment of the probability density. In particular, from the expression of the mean-square displacement, we can determine the type of diffusion process. For $t \ll 1$, the mean-square displacement has power law approximately but the corresponding generalized diffusion coefficient is not a constant but a function of time t. For $t \gg 1$, the diffusion system corresponds basically to localization for the diffusion coefficient $D(t) = D\delta(t)$ and $D(t) = D\,\mathrm{e}^{-t/\tau_c}/\tau_c$, and corresponds to the various types of diffusion process according to the parameter a for $D(t) = Dt^{a-1}/\Gamma(a)$, when the absorbent exists.

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